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An Algebraic Study of Commutation and Termination

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Abstract We study commutation and termination properties in Cohen's ω-algebra; an idempotent semiring with operations for finite and infinite iteration. We provide particularly simple calculational proofs of certain additivity and transformation theorems for termination that depend on commutation, cooperation or simulation properties. We also show that this algebraic approach provides a natural semantics to many standard diagrammatic arguments and that it is especially suited for mechanization in a formal method. Applications are total program correctness, abstract rewriting and concurrency control.
Keywords: program analysis, program semantics, termination analysis, rewriting, Kleene algebra, infinite iteration.

1 Introduction

Termination is an important formal analysis task for programs, state transition systems and term rewrite systems. Many techniques have been developed for reducing termination of a complex system to that of its parts or for transforming it to termination of some more standard system like a wellfounded ordering.

We consider techniques from abstract rewriting; their mechanization and automation. This style of reasoning is traditionally diagrammatic. It is very intuitive, but mechanization requires further formalization, whence a semantics. Predicate logic, of course, may serve this task. But it is rather unspecific: For a smooth translation between diagrams and proofs, additional structure must be developed. Algebra, in particular relational algebra, directly provides such structure. But previous approaches introduced considerable conceptual machinery which is difficult to automate [13,6,7].

Much of abstract rewriting, however, depends solely on the regular operations of relational union, composition and transitive or reflexive-transitive closure. Based on this observation, Kleene algebra has been proposed as a lightweight alternative [16]. It yields a precise relation between rewriting and regular events, supports particularly simple calculational proofs for Church-Rosser theorems and similar statements, provides a simple and natural algebraic semantics for finite rewrite diagrams and is very appropriate for automation. However, it cannot express termination.

Here, therefore, we study Cohen's ω-algebra, an extension of Kleene algebra by an operation for infinite iteration where termination can be expressed simply as absence of infinite iteration. ω-algebra is very suitable for mechanization: Its simple axioms imply a powerful calculus. External inductive arguments considering the length of a rewrite sequence or the number of peaks are replaced by simpler internal fixed-point calculations. There is a strong connection with automata and their decision procedures.

Our case study in termination analysis reveals the potential and limitations of ω-algebra. On the one hand, we obtain many interesting theorems. We prove a very general additivity theorem for termination (a reduction theorem) that subsumes its predecessors [6,1]. We also prove (generalizations of) several transformation theorems [1,10] that depend on commutation, cooperation and simulation properties. All proofs are entirely calculational and mirror precisely the standard diagrammatic arguments. On the other hand, there are some important statements that we cannot handle: roughly, ω-algebra supports only recursion and induction at the end(s) of a sequence, but not in its interior part and therefore models the ω-regular, but not the context-free part of abstract rewriting. In this fragment, however, its conceptual economy imposes a discipline of thought that reveals the essence of proofs: a translation between diagrams and algebra is usually straightforward. Finally, the abstraction to ω-algebra establishes our results in a rich model class beyond relations and rewrite systems. This increases the domain of application.

The remainder of this text is organized as follows. Section 2 introduces to idempotent semirings, Kleene algebra and ω-algebra. Section 3 outlines the elementary calculi of these structures. Section 4 discusses the connection between Kleene algebra and abstract rewriting. Section 5, 6 and 7 introduce and compare different notions of commutation. Section 8, 9 and 10 prove additivity theorems of termination of different levels of generality, corresponding to the different notions of commutation. Section 11 proves transformation theorems for termination based on commutation, simulation and cooperation. Section 12 discusses the limitations of our approach. Section 13 contains a conclusion.

2 Iterator Algebras

A *semiring* is a structure $(A, +, \cdot, 0, 1)$ such that $(A, +, 0)$ is a commutative monoid, $(A, \cdot, 1)$ is a monoid, multiplication distributes over addition from the left and right and zero is a left and right annihilator, that is $a0 = 0 = 0a$ for all $a \in A$ (the operation $\cdot$ is omitted here and in the sequel).

A is an *idempotent semiring* (an *i-semiring*), if addition is idempotent, that is $a + a = a$ for all $a \in A$. Every i-semiring has a natural ordering $\leq$ defined by $a \leq b$ iff $a + b = b$, for all $a, b \in A$. This is the only ordering with least element 0 and for which addition and multiplication are monotone. The additive sub-monoid of an i-semiring is a semilattice with respect to its natural ordering, that is

$$a + b \leq c \Leftrightarrow a \leq c \wedge b \leq c. \tag{1}$$

A *Kleene algebra* [12] is a structure $(K, {}^*)$ such that K is an i-semiring and the *star* * is a unary operation defined by the identities

$$1 + aa^* \leq a^*, \tag{$*$-1}$$
$$1 + a^*a \leq a^*, \tag{$*$-2}$$

and the Horn identities

$$b + ac \leq c \Rightarrow a^*b \leq c, \tag{$*$-3}$$
$$b + ca \leq c \Rightarrow ba^* \leq c, \tag{$*$-4}$$

for all $a, b, c \in K$. This means that a^* is the least prefixed-point of the mappings $\lambda x.ax + b$ and $\lambda x.xa + b$. The star is also monotone with respect to the natural ordering.

Our model of main interest are the set-theoretic relations with addition interpreted as set union, product as relational composition and the star as reflexive transitive closure. Further models are the sets of regular languages (regular events) over some finite alphabet under the regular operations and imperative programs under non-deterministic choice, sequential composition and finite iteration. The class of Kleene algebras is denoted by KA.

For every $K \in \mathsf{KA}$, an operator ${}^+$ for the transitive closure can easily be defined from the star by setting $a^+ = aa^*$. Accordingly, $a^* = 1 + a^+$.

Cohen [3] has extended KA with an operator for (strictly) infinite iteration which is defined as a greatest postfixed-point. An *ω-algebra* is a structure (K, ω) such that $K \in \mathsf{KA}$ and for all $a, b, c \in K$,

$$a^\omega \leq aa^\omega, \tag{ω-1}$$
$$c \leq b + ac \Rightarrow c \leq a^\omega + a^*b. \tag{ω-2}$$

Again, the relational Kleene algebra with infinite iteration is an ω-algebra. We denote the class of ω-algebras by KA_ω.

When we uniformly speak about the star and the omega operator, we call them respectively finite and infinite iterators. We also call ($*$-1), ($*$-2) and (ω-1) *unfold laws* and ($*$-3), ($*$-4) and (ω-2) *induction laws* and *coinduction law*, respectively. Both iterators are uniquely defined.

It is only natural from the relational model to identify termination with the absence of infinite iteration. For $K \in \mathsf{KA}_\omega$ and $a \in K$, we therefore say that *a terminates*, if $a^\omega = 0$. Other popular names for this property are *co-wellfoundedness* or *Noethericity.*

3 Iterator Calculus

This section recalls some useful properties of iterators that appear in calculations of later sections. Most of them are familiar from regular and ω-regular languages [8].

The first statements deal with finite iteration.

Lemma 1. *Let $K \in \mathsf{KA}$. For all $a, b, c \in K$,*

(i) $1 \leq a^*$,
(ii) $aa^* \leq a^* \geq a^*a$,
(iii) $a^*a^* = a^*$,
(iv) $a^i \leq a^*$ *for all* $i \in \mathbb{N}$,
(v) $a^{**} = a^*$,
(vi) $(ab)^*a = a(ba)^*$,
(vii) $(a+b)^* = a^*(ba^*)^*$,
(viii) $a \leq 1 \Rightarrow a^* = 1$,
(ix) $a \leq b \Rightarrow a^* \leq b^*$,
(x) $ac \leq cb \Rightarrow a^*c \leq cb^*$,
(xi) $ca \leq bc \Rightarrow ca^* \leq b^*c$,
(xii) $a^+a^* = a^+ = a^*a^+$,
(xiii) $a^{*+} = a^* = a^{+*}$,
(xiv) $a^{++} = a^+$.

Lemma 2. *Let $t(a_1, \ldots, a_n)$ be a term in the signature of KA in which at most the constants $a_1, \ldots, a_n$ occur. Then for all $K \in \mathsf{KA}$,*

$$K \models t(a_1, \ldots, a_n) \leq (a_1 + \cdots + a_n)^*.$$

Proof. By structural induction. For the base case, obviously, $t \leq (a_1+\cdots+a_n)^*$, where t is one of 0, 1 or $a_1, \ldots, a_n$.

For the induction step, let $t', t'' \leq (a_1 + \cdots + a_n)^*$.

Then $t' + t'' \leq (a_1 + \cdots + a_n)^* + (a_1 + \cdots + a_n)^* = (a_1 + \cdots + a_n)^*$.

Moreover, $t't'' \leq (a_1 + \cdots + a_n)^*(a_1 + \cdots + a_n)^* = (a_1 + \cdots + a_n)^*$, since $a^*a^* = a^*$.

Finally, $t'^* \leq (a_1 + \cdots + a_n)^{**} = (a_1 + \cdots + a_n)^*$, by monotonicity of star and $a^{**} = a^*$. □

The following statements deal with infinite iteration. Some of them can be found in [3]. We provide complete proofs since we did not find such proofs elsewhere in the literature.

The first lemma concerns 0^ω and 1^ω.

Lemma 3. *Let $K \in \mathsf{KA}_\omega$.*

(i) $0^\omega = 0$.
(ii) 1^ω *is the greatest element of* K.

Proof. (i) $0^\omega \leq 00^\omega = 0$.

(ii) Setting $a = 1$ and $b = 0$ in (ω-2) yields $c \leq 1^\omega$ for all c, which means that 1^ω is the greatest element of the algebra. □

Lemma 4. *Let $K \in \mathsf{KA}_\omega$. For all $a, b \in K$,*

(i) $a^\omega = aa^\omega$,
(ii) $a^\omega b \leq a^\omega$,

(iii) $a^+a^\omega = a^\omega$,
(iv) $a^*a^\omega = a^\omega$,
(v) $a \leq a^\omega$, *if* a *is dense, that is* $a \leq aa$.

Proof. (i) Obviously, $a^\omega \leq aa^\omega$, by (ω-1). For the converse inequality it suffices, by (ω-2), to show that $aa^\omega \leq aaa^\omega$. This holds, since by ($\omega$-1) $a^\omega \leq aa^\omega$.

(ii) By (ω-2), it suffices to show that $a^\omega b \leq aa^\omega b$. But this is the case by (ω-1).

(iii) For $a^+a^\omega \leq a^\omega$, it suffices by ($\omega$-2) that $a^+a^\omega \leq aa^+a^\omega$. But $a^+a^\omega \leq a^+aa^\omega = aa^+a^\omega$ by (ω-1). For the converse direction, $a^\omega = aa^\omega \leq a^+a^\omega$.

(iv) Similar to (iii).

(v) For $a \leq a^\omega$ it suffices to show that $a \leq aa$ by (ω-2), whence density. □

Lemma 5. *Let* $K \in \mathsf{KA}_\omega$. *For all* $a, b \in K$,

$$a \leq b \Rightarrow a^\omega \leq b^\omega.$$

Proof. Let $a \leq b$. For $a^\omega \leq b^\omega$ it suffices by (ω-2) to show that $a^\omega \leq ba^\omega$. Using (ω-1) we calculate $a^\omega \leq aa^\omega \leq ba^\omega$. □

Corollary 6. *Let* $K \in \mathsf{KA}_\omega$. *For all* $a \in K$,

$$a \geq 1 \Rightarrow a^\omega = 1^\omega.$$

Lemma 7. *Let* $K \in \mathsf{KA}_\omega$. *For all* $a \in K$,

(i) $a^{+\omega} = a^\omega$,
(ii) $a^{\omega+} = a^\omega$,
(iii) $a^{*\omega} = 1^\omega$,
(iv) $a^{\omega*} = a^\omega$,
(v) $a^{\omega\omega} \leq a^\omega$.

Proof. (i) $a^\omega \leq a^{+\omega}$ follows from monotonicity of $^\omega$ and $a \leq a^+$. For the converse direction it suffices by (ω-2) to show that $a^{+\omega} \leq aa^{+\omega}$. We calculate

$$a^{+\omega} \leq a^+a^{+\omega} = aa^*a^{+\omega} = aa^{+*}a^{+\omega} = aa^{+\omega}.$$

The third step follows by Lemma 1 (xiii). The last step uses Lemma 4 (iv).

(ii) $a^\omega \leq a^{\omega+}$ follows from $b \leq b^+$. For the converse direction, it suffices by (ω-2) to show that $a^{\omega+} \leq aa^{\omega+}$. We calculate

$$a^{\omega+} = a^\omega a^{\omega*} \leq aa^\omega a^{\omega*} = aa^{\omega+},$$

using the definition of a^+ in the first and third step and (ω-1) in the second step.

(iii) Immediate from Corollary 6.

(iv) $a^\omega \leq a^{\omega*}$, since $b \leq b^*$. For the converse direction, we calculate

$$a^{\omega*} = a^\omega a^{\omega+} = a^\omega a^\omega \leq a^\omega.$$

The first step uses the definition of a^+. The second step uses (iii). The third step uses Lemma 4) (vii).

(v) $a^{\omega\omega} \leq a^\omega a^{\omega\omega} \leq a^\omega$ by (ω-1) and Lemma 4 (ii). □

Lemma 8. *Let $K \in \mathsf{KA}_\omega$. For all $a, b, c \in K$,*

$$ab \leq ca \Rightarrow ab^\omega \leq c^\omega.$$

Proof. By (ω-2), it suffices to show that $ab^\omega \leq cab^\omega$. We calculate

$$ab^\omega = abb^\omega \leq cab^\omega.$$

$\square$

Lemma 9. *Let $K \in \mathsf{KA}_\omega$ and let $a, b \in K$. Then*

$$(a+b)^\omega = (a^*b)^\omega + (a^*b)^*a^\omega.$$

Proof. We first show that $(a^*b)^\omega + (a^*b)^*a^\omega \leq (a+b)^\omega$. We calculate

$$\begin{aligned}(a^*b)^\omega + (a^*b)^*a^\omega &\leq (a+b)^{+\omega} + (a+b)^*(a+b)^\omega \\ &= (a+b)^\omega + (a+b)^\omega \\ &= (a+b)^\omega\end{aligned}$$

The first step holds, since $a^*b \leq (a+b)^+$, $(a^*b)^* \leq (a+b)^*$ and by monotonicity of ω. The second step follows from Lemma 7 (ii) and Lemma 4 (iv).

For the converse direction it suffices to show by (ω-2) that

$$(a+b)^\omega \leq (a^*b)(a+b)^\omega + a^\omega.$$

and

$$(a+b)^\omega \leq a(a+b)^\omega + b(a+b)^\omega = (a+b)(a+b)^\omega,$$

applying (ω-2) a second time. But this holds by (ω-1). $\square$

4 Kleene Algebra and Abstract Rewriting

An *abstract rewrite system* is a set together with a family of binary relations (c.f. [11]). Properties of abstract rewrite systems belong to the foundations of term rewriting, of the λ-calculus and of functional programming. These are, for instance, Church-Rosser and confluence properties, but also commutation properties. Traditionally, the latter have been studied first of all for proving termination of rewrite systems, for establishing properties of combinations of rewrite systems or for proving properties of recursive programs. More recently, an approach to rewriting for pre-congruences has been strongly based on commutation [14,15]. This is also interesting for studying concurrent systems.

Here, we investigate abstract rewriting even more abstractly as an ω-algebra. This is possible, since set-theoretic relations under the (ω-)regular operations are models of iterator algebras. The transition from relations to Kleene algebra can be motivated by the following simple example. Let R and S be binary relations over some set A. As usual in rewriting, we write $a \to_R b$ instead of $(a, b) \in R$ for all $a, b \in A$. Consider the following simple semi-commutation property which holds, if for all $a, b, c \in A$, if $a \to_S b$ and $b \to_R c$, then there is some $d \in A$ such that $a \to_R d$ and $d \to_S c$. In concurrency theory, this property expresses the fact that execution of R may always be given priority over execution of S. It is very common in rewriting to express this by a semi-commuting diagram such as the following.

$$\begin{tikzcd} a \arrow[r, "S"] \arrow[d, dashed, "R"'] & b \arrow[d, "R"] \\ d \arrow[r, dashed, "S"'] & c \end{tikzcd}$$

Here, the solid lines denote the universally quantified, the dashed lines the existentially quantified part of the diagram. A first step of abstraction is the elimination of points. In a more compact point-free style, the above semi-commutation property can be written as $S \circ R \subseteq R \circ S$. But expressions of this form now are accessible by Kleene algebra. The semi-commutation property then becomes $ba \leq ab$. More generally, Kleene algebra gives a rigorous algebraic semantics to rewriting diagrams. See [16] for a discussion. See also [9] for related diagrams for allegories. Composition of diagrams, for instance, is modeled at the level of Kleene algebra by monotonicity properties.

Analogously, the Church-Rosser property of abstract rewriting now translates to $(a + b)^* \leq a^*b^*$ in Kleene algebra, where b stands for the converse of a and confluence becomes $b^*a^* \leq a^*b^*$. It has been shown in [16] that Kleene algebra admits in particular point-free calculational proofs of Church-Rosser theorems and similar statements. To show that confluence implies the Church-Rosser property in Kleene algebra amounts to proving that the Horn identity

$$b^*a^* \leq a^*b^* \Rightarrow (a + b)^* \leq a^*b^*. \tag{2}$$

is a theorem of KA. It suffices to show that the assumption $b^*a^* \leq a^*b^*$ implies $a^*(ba^*)^* \leq a^*b^*$ by Lemma 1 (vii) and therefore $a^* + a^*b^*ba^* \leq a^*b^*$ by ($*$-4). We calculate

$$a^* + a^*b^*ba^* \leq a^* + a^*b^*a^* \leq a^*a^*b^* \leq a^*b^* + a^*b^* = a^*b^*.$$

The first step uses Lemma 1 (ii). The second step uses the assumption, the third step uses Lemma 1 (i) and (iii). The fourth step uses idempotency of addition.

While similar proofs have been used in [16] for dealing with that part of abstract rewriting in Kleene algebra that does not mention termination, we now use ω-algebra in calculational proofs in presence of termination properties. By this abstraction, previous results then hold for a larger class of models. The main interest of the algebraic approach is as follows. Proofs in abstract rewriting are usually diagrammatic such as the above. We simulate them by purely calculational, whence induction-free proofs that nevertheless precisely mirror the diagrammatic ones. This is in contrast to other formal approaches that are based on complex formalizations of finite and infinite sequences and their interactions.

Our approach is therefore particularly suited for mechanization, which is very important for termination analysis in the context of a formal method.

5 Type-1 Commutation and Termination

We now introduce two notions of commutation that already appear in [1] and in [15].

Let $A \in \mathsf{KA}$ and let $a, b \in A$. We say that a *(type-1) semi-commutes* over b, if $ba \leq a^+b^*$. a *(type-1) quasi-commutes* over b, if $ba \leq a(a+b)^*$. We write $sc_1(a,b)$ if a type-1 semi-commutes over b and $qc_1(a,b)$, iff a type-1 quasi-commutes over b.

Both commutation properties have in common that a is commuted in some sense to the left of b, although in a less trivial way than by bubble sort and in the example of the previous section. When it is obvious from the context, we just speak about semi-commutation and quasi-commutation.

We now compare the two commutation properties. But doing this first requires a technical lemma. It splits an arbitrary sequence of a- and b-steps into a "good" sequence where all a-steps are to the left of b-steps and into a "bad" sequence that contains reversals of the form ba.

Lemma 10. *Let $A \in \mathsf{KA}$. For all $a, b \in A$,*

$$(a+b)^* = a^*b^* + a^*b^+a(a+b)^*. \tag{3}$$

Proof. First, $a^*b^* + a^*b^+a(a+b)^* \leq (a+b)^*$ holds by Lemma 2.

For the converse direction, it suffices by ($*$-4) to show that

$$(a^*b^* + a^*b^+a(a+b)^*)(a+b) \leq a^*b^*a^*b^+a(a+b)^*.$$

We calculate

$$\begin{aligned}(a^*b^* + a^*b^+a(a+b)^*)(a+b) &= a^*b^*a + ab^* + a^*b^+a(a+b)^+\\ &= a^*(1+b^+)a + ab^* + a^*b^+a(a+b)^+\\ &= a^+ + ab^* + a^*b^+a(1+(a+b)^+)\\ &\leq a^*b^* + a^*b^+a(a+b)^*.\end{aligned}$$

□

Using Lemma 10, we can now use semi-commutation and quasi-commutation for replacing bad sequences. But since there is transitive closure involved, we use the following lemma to extend semi-commutation and quasi-commutation to transitive and reflexive transitive closures.

Lemma 11. *Let $A \in \mathsf{KA}$. For all $a, b \in A$,*

(i) $qc_1(a,b) \Leftrightarrow b^*a \leq a(a+b)^*$,
(ii) $qc_1(a,b) \Leftrightarrow b^+a \leq a(a+b)^*$,

(iii) $sc_1(a,b) \Leftrightarrow b^*a \leq a^+b^*$,
(iv) $sc_1(a,b) \Leftrightarrow b^+a \leq a^+b^*$,

Proof. (i) Let $b^*a \leq a(a+b)^*$. Using $b \leq b^*$, we calculate $ba \leq b^*a \leq a(a+b)^*$.
Let $ba \leq a(a+b)^*$. Then $b^*a \leq a(a+b^*)^* \leq a(a+b)^*$ follows from Lemma 2.
(ii) Immediate from (i).
(iii) Let $b^*a \leq a^+b^*$. Using $b \leq b^*$, we calculate $ba \leq b^*a \leq a^+b^*$.
Let $ba \leq a^+b^*$. Then $b^*a \leq a^+b^{**} = a^+b^*$ follows from $b^{**} = b^*$.
(iv) Immediate from (iii). □

The following lemma compares semi-commutation with quasi-commutation. A relational variant has been used in [15] in the context of rewriting for pre-congruences.

Lemma 12. *Let $A \in \mathsf{KA}_\omega$. For all $a, b \in A$ with $a^\omega \leq 0$,*

$$qc_1(a,b) \Leftrightarrow sc_1(a,b).$$

Proof. Let a semi-commute over b. Then $ba \leq a^+b^* \leq a(a+b)^*$ by Lemma 2.
Let a quasi-commute over b, that is $ba \leq a(a+b)^*$ and let $a^\omega = 0$. We show that $a(a+b)^* \leq a^+b^*$. First, we calculate

$$\begin{aligned} a(a+b)^* &= a(a^*b^* + a^*b^+a(a+b)^*) \\ &= a^+b^* + a^+b^+a(a+b)^* \\ &\leq a^+b^* + a^+a(a+b)^*(a+b)^* \\ &= a^+b^* + a^+a(a+b)^*. \end{aligned}$$

The first step uses Lemma 10. The third step uses Lemma 11 (ii). The fourth step uses $a^*a^* = a^*$.

The resulting inequality is of the form $X \leq s + tX$. We can therefore apply (ω-2) and obtain

$$a(a+b)^* \leq a^{+\omega} + a^{+*}a^+b^* = a^{+\omega} + a^+b^*,$$

using Lemma 1 (xiii) and (xii) for further simplification. Now the claim follows immediately from Lemma 7 (ii), that is $a^{+\omega} = a^\omega$, and the assumption $a^\omega = 0$. □

Let us compare the calculational proof of Lemma 12 with a previous diagrammatic one. A standard trick for reasoning in presence of termination is reductio ad absurdum, which amounts to constructing an infinite sequence. We exploit the correspondence between semi-commutation properties and rewrite diagrams. It turns out that the algebraic proof can be reconstructed directly from the diagrams. In this sense, KA_ω captures precisely the standard intuitive arguments of abstract rewriting with termination assumptions, but in a formally rigorous calculational style. Therefore, it yields a natural semantics for rewrite diagrams.

Reconsider the non-trivial part of the proof of Lemma 12 in the relational model, revisiting the proof in [15]. We assume by reductio ad absurdum that

a quasi-commutes over b and that a does not semi-commute over b, which is equivalent to $b^+a \leq a(a+b)^*$ and $b^+a \not\leq a^+b^*$ by Lemma 11. In the relational model one would usually verify these equivalences by induction in the length of the initial b-sequence. Then the assumptions imply that the bad sequence containing b^+a must be of the form $a^+b^+a(a+b)^*$. This has now been formally shown in Lemma 2. Since this sequence contains again a subsequence of the form b^+a, we can iterate the replacement and obtain the following Jacob's ladder.

$$
\begin{tikzcd}
{} & & {} \\
\cdot \arrow[u, dotted, no head] \arrow[r, "b^+"'] & \cdot \arrow[r, "a"'] & \cdot \arrow[u, dotted, no head] \\
\cdot \arrow[u, "a^+"] \arrow[r, "b^+"'] & \cdot \arrow[r, "a"'] & \cdot \arrow[u, "(a+b)^*"'] \\
\cdot \arrow[u, "a^+"] \arrow[r, "b^+"'] & \cdot \arrow[r, "a"'] & \cdot \arrow[u, "(a+b)^*"']
\end{tikzcd}
$$

Obviously, the left-hand side of the ladder is an infinite a-sequence contradicting our termination assumption.

Compared to our calculational proof, there are three main differences. First, we reason by contraposition. But in this case, this makes no difference to the direct proof, where instead of observing an infinite sequence, we prohibit it. Second, and instead, we prohibit the good sequences of the form a^+b^*, whereas we carry that part with us in the calculational proof. Third, we perform an iteration using the formula $X = s + tX$ instead of using the coinduction rule (ω-2) which carries us in one step to the limit of this iteration. In the relational model, this is no problem, since the limit exists and is equal to the result of the iteration.

To sum up, we see that our calculational proof precisely reconstructs the previous diagrammatical argument from [15]. A similar reconstruction has been done previously for the diagrams of Church-Rosser proof and related statements that do not use termination assumptions in [16]. In all these cases, our algebraic proofs capture precisely the intuition behind the diagrams. Moreover, many steps that are usually left implicit are now accessible to a simple formalization. This is in particular interesting for mechanization.

6 Type-2 Commutation and Termination

We now introduce two more general notions of commutation and prove a slightly weaker variant of Lemma 12. We say that *a type-2 semi-commutes* over b, if $ba \leq a^+b^* + b^+$. This notion has been called *weak commutation* in [15] and investigated also in [16]. It is the appropriate notion for studying Church-Rosser properties of non-symmetric transitive relations and for developing Knuth-Bendix completion procedures for this class. We say that *a type-2 quasi-commutes* over b, if $ba \leq a(a+b)^* + b$. We write $sc_2(a,b)$, if a type-2 semi-commutes over b and $qc_2(a,b)$, if a type-2 quasi-commutes over b. The latter property has been used already in [6].

The following lemma yields a more symmetric form of right-hand side of type-2 semi-commutation.

Lemma 13. *Let $A \in \mathsf{KA}$. For all $a, b \in A$, $a^+b^* + b^+ = a^+b^* + a^*b^+$.*

Proof. Obviously, $a^+b^* + b^+ \leq a^+b^* + a^*b^+$, since $1 \leq a^*$.

For the converse direction, we must show that

$$a^+b^* \leq a^+b^* + b^+, \qquad a^*b^+ \leq a^+b^* + b^+.$$

The first inequality being trivial, it remains to consider the second one. By ($*$-3), it suffices to show that

$$a(a^+b^* + b^+) + b^+ \leq a^+b^* + b^+,$$

whence

$$aa^+b^* \leq a^+b^* + b^+ \qquad ab^+ \leq a^+b^* + b^+, \qquad b^+ \leq a^+b^* + b^+.$$

The first inequality holds, since $aa^+ \leq a^+$. The second inequality holds, since $a \leq a^+$ and $b^+ \leq b^*$. The third inequality holds, since $b \leq b^+$. $\square$

We obtain the following variant of Lemma 11.

Lemma 14. *Let $A \in \mathsf{KA}$. For all $a, b \in A$,*

(i) $qc_2(a,b) \Rightarrow b^*a \leq a(a+b)^* + b^+$,
(ii) $qc_2(a,b) \Rightarrow b^+a \leq a(a+b)^* + b^+$,
(iii) $sc_2(a,b) \Leftrightarrow b^*a \leq a^+b^* + b^+$,
(iv) $sc_2(a,b) \Leftrightarrow b^+a \leq a^+b^* + b^+$.

Proof. (i) $b^*a = (1+b^+)a = a + b^+a$. Using (ii) yields $b^*a \leq a + a(a+b)^* + b^+ = a(a+b)^* + b^+$, since $a \leq a(a+b)^*$.

(ii) Let $ba \leq a(a+b)^* + b$. Then $b^+a \leq a(a+b^+)^* + b^+ \leq a(a+b)^* + b^+$, using Lemma 2.

(iii) Let $b^*a \leq a^+b^* + b^+$. Using $b \leq b^*$, we calculate $ba \leq b^*a \leq a^+b^* + b^+$.

Let $ba \leq a^+b^* + b^+$. Obviously, $b^*a = (1+b^+)a = a + b^+a$. Using (iv) yields $b^*a \leq a + a^+b^* + b^+ = a^+b^* + b^+$, since $a \leq a^+b^*$.

(iv) Let $b^*a \leq a^+b^* + b^+$. The proof is analogous to (iii), using$b \leq b^+$.

Let $ba \leq a^+b^* + b^+$. Then $b^+a \leq a^+b^{+*} + b^{++} = a^+b^* + b^+$, using Lemma 1 (xiii) and (xiv). $\square$

We also have the following obvious relation between type-1 and type-2 commutation.

Lemma 15. *Let $A \in \mathsf{KA}$. For all $a, b \in A$,*

(i) $qc_1(a, b) \Rightarrow qc_2(a, b)$,
(ii) $sc_1(a, b) \Rightarrow sc_2(a, b)$.

The following corollary from [15] is immediate from Lemma 12 and Lemma 15 (ii).

Corollary 16. *Let $A \in \mathsf{KA}_\omega$. For all $a, b \in A$ with $a^\omega \leq 0$,*

$$qc_1(a, b) \Rightarrow sc_2(a, b).$$

Lemma 17. *Let $A \in \mathsf{KA}$. For all $a, b \in A$,*

(i) $sc_2(a, b) \Rightarrow ba^* \leq a^+b^* + b^+$,
(ii) $qc_2(a, b) \Rightarrow ba^* \leq a(a + b)^* + b$,
(iii) $qc_2(a, b) \Rightarrow a(a + b)^* + b^+ = (a + b^+)(a + b)^*$.

Proof. (i) By Lemma 13, $a^+b^* + b^+ = a^+b^* + a^*b^+$. Then the claim follows by symmetry from Lemma 14 (iii).

(ii) Let $ba \leq a(a + b)^* + b$. By star induction, it suffices to show that

$$(a(a + b)^* + b)a + b \leq a(a + b)^* + b.$$

We calculate

$$\begin{aligned}(a(a + b)^* + b)a + b &= a(a + b)^*a + ba + b \\ &\leq a(a + b)^* + a(a + b)^* + b + b \\ &= a(a + b)^* + b.\end{aligned}$$

The second step uses Lemma 2, Lemma 1 (iii) and the assumption.

(iii) For $a(a + b)^* + b^+ \leq (a + b^+)(a + b)^*$, we calculate

$$a(a + b)^* + b^+ \leq a(a + b)^* + b^+(a + b)^* = (a + b^+)(a + b)^*,$$

using $1 \leq (a + b)^*$.

For the converse direction, it remains to show that $b^+(a+b)^* \leq a(a+b)^*+b^+$. By star induction, it suffices to show that

$$(a(a + b)^* + b^+)(a + b) + b^+ \leq a(a + b)^* + b^+.$$

We calculate

$$\begin{aligned}(a(a + b)^* + b^+)(a + b) + b^+ &= a(a + b)^*(a + b) + b^+a + b^+b + b^+ \\ &\leq a(a + b)^* + a(a + b)^* + b + b^+ \\ &= a(a + b)^* + b^+.\end{aligned}$$

The second step uses Lemma 1 (ii), $b^+b \leq b^+$ and Lemma 14 (ii) together with the assumption. The third step uses $a \leq a^+$. ∎

Corollary 16 and Lemma 12 can be generalized as follows.

Lemma 18. *Let $A \in \mathsf{KA}_\omega$. For all $a, b \in A$ with $a^\omega \le 0$,*

$$qc_2(a,b) \Rightarrow sc_2(a,b).$$

Proof. The proof is analogous to that of Lemma 12. Let $a^\omega \le 0$ and $ba \le a(a+b)^* + b$. We show that $a(a+b)^* + b \le a^+b^* + b^+$. First, we calculate

$$\begin{aligned} a(a+b)^* + b^+ &= a(a^*b^* + a^*b^+a(a+b)^*) + b^+ \\ &= a^+b^* + a^+b^+a(a+b)^* + b^+ \\ &\le a^+b^* + a^+(a(a+b)^* + b^+)(a+b)^* + b^+ \\ &= a^+b^* + a^+(a+b^+)(a+b)^* + b^+ \\ &= a^+b^* + b^+ + a^+(a(a+b)^* + b^+). \end{aligned}$$

The first step uses Lemma 2. The third step uses the assumption of quasi-commutation. The fourth step uses distributivity and Lemma 1 (iii). The fifth step uses Lemma 17 (iii) together with the assumption of quasi-commutation.

The resulting inequality is of the form $X \le s + tX$. We can therefore apply (ω-2) and obtain

$$\begin{aligned} a(a+b)^* + b^+ &\le a^{+\omega} + a^{+*}(a^+b^* + b^+) \\ &= a^{+\omega} + a^+b^* + a^*b^+ \\ &= a^{+\omega} + a^+b^* + b^+. \end{aligned}$$

The second step uses distributivity and Lemma 1 (xiii) and (xii). The third step uses Lemma 13.

Now the claim follows from $b \le b^+$, Lemma 7 (ii) and the assumption $a^\omega = 0$. $\square$

We see that Lemma 18 is a rather straightforward generalization of Lemma 12. However, it is an implication, whereas for type-1 commutation properties a bi-implication holds. This symmetry is restored in the following section.

7 Type-3 Commutation and Termination

A careful inspection of the proofs of the previous section leads to the question, why not use the more general quasi-commutation property $ba \le a(a+b)^* + b^+$ instead of type-2 quasi-commutation. We call this third property *type-3 quasi-commutation*. We write $qc_3(a,b)$ if a type-3 quasi-commutes over b.

We have the following variant of Lemma 14

Lemma 19. *Let $A \in \mathsf{KA}$. For all $a, b \in A$,*

(i) $qc_2(a,b) \Leftrightarrow b^*a \le a(a+b)^* + b^+$,
(ii) $qc_2(a,b) \Leftrightarrow b^+a \le a(a+b)^* + b^+$.

Proof. (i) Let $b^*a \leq a(a+b)^* + b^+$. The proof is analogous to that of Lemma 11 (i).

Let $ba \leq a(a+b)^* + b^+$. The proof is analogous to that of Lemma 14 (i).

(ii) Let $b^+a \leq a(a+b)^* + b^+$. The proof is analogous to that of Lemma 11 (ii).

Let $ba \leq a(a+b)^* + b^+$. Then $b^+a \leq a(a+b^+)^* + b^{++} = a(a+b)^* + b^+$, using Lemma 2 and Lemma 1 (xiv). □

Lemma 19 restores the symmetry of Lemma 11 that was absent in Lemma 14.

Lemma 20. *Let $A \in \mathsf{KA}$. For all $a, b \in A$,*

$$qc_2(a,b) \Rightarrow qc_3(a,b).$$

A careful inspection of the proof of Lemma 17 (iii) shows that the respective property also holds for type-3 quasi-commutation.

Lemma 21. *Let $A \in \mathsf{KA}$. For all $a, b \in A$,*

$$qc_2(a,b) \Rightarrow a(a+b)^* + b^+ = (a+b^+)(a+b)^*. \tag{4}$$

The following Lemma restores the symmetry of Lemma 12 that has been absent in Lemma 18.

Lemma 22. *Let $A \in \mathsf{KA}_\omega$. For all $a, b \in A$ with $a^\omega \leq 0$,*

$$qc_3(a,b) \Leftrightarrow sc_2(a,b).$$

Proof. Let a type-2 semi-commute over b. Then $ba \leq a^+b^* + b^+ \leq a(a+b)^* + b^+$ by Lemma 2.

Let a type-3 quasi-commute over b. The proof then follows from a careful inspection of the proof of Lemma 18, using Lemma 21 instead of Lemma 17 (iii). □

8 Type-1 Commutation and Additivity of Termination

We now prove an abstract algebraic counterpart of the quasi-commutation lemma of Bachmair and Dershowitz [1], which says that termination of relations is preserved by unions, if the relations quasi-commute. More precisely, this section is concerned with type-1 semi-commutation and quasi-commutation.

We first show a simple technical lemma.

Lemma 23. *Let $A \in \mathsf{KA}_\omega$. For all $a, b \in A$, $b^*(b^*a)^\omega = (b^*a)^\omega$.*

Proof. We calculate

$$(b^*a)^\omega = b^*a(b^*a)^\omega = b^*b^*a(b^*a)^\omega = b^*(b^*a)^\omega.$$

The first step and the last step use Lemma 4 (i). The second step uses Lemma 1 (iii). □

Lemma 24. *Let $A \in \mathsf{KA}_\omega$.*

(i) Let $a, b \in A$ and $a^\omega \leq 0$. Then

$$qc_1(a,b) \Leftrightarrow b^*a \leq a^+b^*.$$

(ii) Let $a, b \in A$ and let $qc_1(a,b)$. Then

$$a^\omega \leq 0 \Rightarrow (b^*a)^\omega \leq 0.$$

Proof. (i) Immediate from Lemma 12 and Lemma 11 (iii).

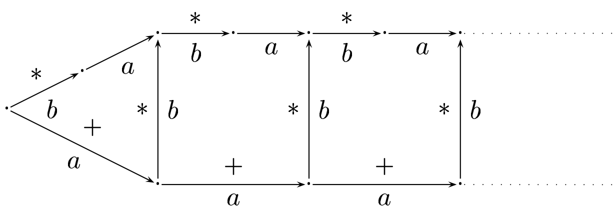
(ii) Let $a^\omega \leq 0$ and let $qc_1(a,b)$. We calculate

$$(b^*a)^\omega = b^*a(b^*a)^\omega \leq a^+b^*(b^*a)^\omega = a^+(b^*a)^\omega.$$

The first step uses Lemma 4. The second step uses (i). The third step uses Lemma 23. But $(b^*a)^\omega \leq a^+(b^*a)^\omega$ implies $(b^*a)^\omega \leq a^{+\omega}$ by omega coinduction, from which the claim $(b^*a)^\omega \leq 0$ follows by Lemma 7 (i) and the termination assumption. □

Lemma 24 (ii) is roughly Lemma 2 of [1].

At this stage we again compare our calculational proof with a diagrammatic one from [15]. A diagrammatic proof of Lemma 24 (ii) is as follows. Assume an infinite b^*a-chain for reductio ad absurdum. By Lemma 12 and Lemma 11 (iii), which have now been proved formally, every b^*a-sequence can be transformed into a a^+b^*-chain, if a terminates and quasi-commutes over b. Thus the diagram



yields an infinite a-sequence, a contradiction. Again, our calculational proofs are direct reconstructions of this diagrammatic argument.

We now prove the quasi-commutation theorem of Bachmair and Dershowitz (Theorem 1 of [1]).

Theorem 25. *Let $A \in \mathsf{KA}_\omega$. Let $a, b \in A$ and let $qc_1(a,b)$. Then $a+b$ terminates iff a and b terminate.*

$$a^\omega + b^\omega \leq 0 \Leftrightarrow (a+b)^\omega \leq 0.$$

Proof. Let $(a+b)^\omega \leq 0$ and let $qc_1(a,b)$. Then $a^\omega + b^\omega \leq 0$ follows from monotonicity and (1).

Let $a^\omega + b^\omega \leq 0$. By Lemma 9,

$$(a+b)^\omega = (b^*a)^\omega + (b^*a)^*b^\omega \leq 0.$$

$(b^*a)^\omega$ vanishes by Lemma 24 (ii), using the assumption of quasi-commutation and termination of a. $(b^*a)^*b^\omega$ vanishes by termination of b. □

Proposition 26. *Let $A \in \mathsf{KA}_\omega$. Let $a,b,c \in A$ and let $qc_1(c^*a, c^*b)$. Then*

$$(c^*a)^\omega + (c^*b)^\omega \leq 0 \Leftrightarrow (c^*(a+b))^\omega \leq 0.$$

Proof. Obviously, $c^*(a+b) = c^*a + c^*b$. Then apply Theorem 25. □

Theorem 25 and its corollary have interesting applications in term rewriting. See [1] for a discussion and further results.

9 Type-2 Commutation and Additivity of Termination

We now prove the quasi-commutation theorem of Doornbos, Backhouse and van der Woude [6]. This result generalizes that of Bachmair and Dershowitz. It is concerned with type-2 semi-commutation and quasi-commutation. While [6] used a relational fixed-point calculus, we use the leaner machinery of ω-algebra. Moreover, our proof is closer to the traditional diagrammatic proofs and therefore perhaps more intuitive.

First, we prove a weak variant of Lemma 24 (i) and (ii).

Lemma 27. *Let $A \in \mathsf{KA}_\omega$.*

(i) Let $a,b \in A$ and $a^\omega \leq 0$. Then

$$qc_2(a,b) \Rightarrow b^*a \leq a^+b^* + b^+.$$

(ii) Let $a,b \in A$ and let $qc_2(a,b)$. Then

$$a^\omega + b^\omega \leq 0 \Rightarrow (b^*a)^\omega \leq 0.$$

Proof. (i) Immediate from Lemma 18 and Lemma 14 (iii).

(ii) Let $a^\omega \leq 0$ and let $qc_2(a,b)$. We calculate

$$(b^*a)^\omega = b^*a(b^*a)^\omega \leq (a^+b^* + b^+)(b^*a)^\omega = (a^+b)(b^* + a)^\omega \leq (a+b)(b^*a)^\omega.$$

The first step uses Lemma 4 (i). The second step uses (i). The third step uses Lemma 23. The fourth step uses $a^+ = aa^* \leq a(b^*a)^*$ and then Lemma 4 (iv). Therefore, by omega coinduction,

$$(b^*a)^\omega \leq a^\omega + a^*b(b^*a)^\omega. \tag{5}$$

We now consider $b(b^*a)^\omega$. We calculate

$$\begin{aligned} b(b^*a)^\omega &\leq b(a^\omega + a^*b(b^*a)^\omega \\ &= ba^\omega + ba^*b(b^*a)^\omega \\ &\leq ba^\omega + (a^+b + b^+)b(b^*a)^\omega \\ &\leq ba^\omega + a^+b(b^*a)^\omega + bb(b^*a)^\omega . \end{aligned}$$

The first step inserts (5). The third step uses Lemma 17 (i). The fourth step uses $b^+ = b^*b$ and Lemma 23.

But, since obviously $(b^*a)^\omega \not\leq a^+c$ for all $c \in A$, the inequality $b(b^*a)^\omega \leq ba^\omega + a^+b(b^*a)^\omega + bb(b^*a)^\omega$ holds iff

$$b(b^*a)^\omega \leq ba^\omega + a^+b(b^*a)^\omega + bb(b^*a)^\omega$$

holds, whence, by omega coinduction,

$$b(b^*a)^\omega \leq b^\omega + bb^*a^\omega = b^\omega + b^+a^\omega .$$

Insertion into (5) yields

$$(b^*a)^\omega \leq a^\omega + a^*(b^\omega + b^+a^\omega).$$

Therefore, with $a^\omega \leq 0$ and $b^\omega \leq 0$ also $(b^*a)^\omega \leq 0$. □

Theorem 28. *Let $A \in \mathsf{KA}_\omega$. Let $a, b \in A$ and let $qc_2(a, b)$, Then $a+b$ terminates iff a and b terminate.*

$$a^\omega + b^\omega \leq 0 \Leftrightarrow (a + b)^\omega \leq 0.$$

Proof. Along the lines of the proof of Theorem 25, but using Lemma 27 (ii) instead of Lemma 24 (ii). □

This theorem encompasses some other conditions [6].

Corollary 29. *Let $A \in \mathsf{KA}_\omega$. For all $a, b \in A$, $a + b$ terminates, iff a and b terminate, whenever one of the following conditions hold.*

(i) $qc_1(a, b)$,
(ii) $a + b$ *is transitive,*
(iii) $ba \leq b$.

Proof. (i) By Lemma 15 (i), $qc_1(a, b) \Rightarrow qc_2(a, b)$.
(ii) $(a + b)(a + b) \leq (a + b) \Rightarrow ba \leq a + b \leq a(a + b)^* + b^+$.
(iii) $ba \leq b \leq b \leq a(a + b)^* + b$. □

Again, the calculational proofs can be visualized diagrammatically.

10 Type-3 Commutation and Additivity of Termination

In this section we further generalize the theorem of Doornbos, Backhouse and van der Woude to type-3 commutation.

First, we prove a weaker variant of Lemma 27 (i) and (ii).

Lemma 30. *Let $A \in \mathsf{KA}_\omega$.*

(i) Let $a, b \in A$ and $a^\omega \leq 0$. Then

$$qc_3(a,b) \Rightarrow b^*a \leq a^+b^* + b^+.$$

(ii) Let $a, b \in A$ and let $qc_3(a,b)$. Then

$$a^\omega \leq 0 \wedge b^\omega \leq 0 \Rightarrow (b^*a)^\omega \leq 0.$$

Proof. The proofs follow immediately from a careful analysis of those of Lemma 27. □

Consequently, we have the following theorem.

Theorem 31. *Let $A \in \mathsf{KA}_\omega$. Let $a, b \in A$ and let $qc_3(a,b)$, Then $a+b$ terminates iff a and b terminate.*

$$a^\omega + b^\omega \leq 0 \Leftrightarrow (a+b)^\omega \leq 0.$$

11 Termination in Presence of Further Properties

Bachmair and Dershowitz also present a series of transformation theorems to infer termination of some complex relation from termination of some more standard one. Their statements use commutation properties together with an embedding function of the complex ordering into the simpler one. Such embeddings into a standard ordering, like for instance the natural numbers, are a common technique in termination analysis. Here, we use a more general approach with a simulation relation instead of an embedding, as common in the analysis of concurrent systems and state transition systems. Moreover, we relax certain assumptions that are not substantial for the proof.

For $A \in \mathsf{KA}$ and $a, b \in A$ we say that b *simulates* a, iff $as \leq sb$ for some $s \in A$. In particular, by Lemma 1 (x), simulation implies that $a^*s \leq sb^*$, as expected.

The following statement generalizes Theorem 4 of [1].

Theorem 32. *Let $A \in \mathsf{KA}_\omega$. For all $a, b, c, s, t_1, t_2 \in A$, let $a \leq scs$, $sbs \leq t_1t_2$, $t_2t_1 \leq 1$, and $t_2c \leq ct_2$. Then*

$$c^\omega \leq 0 \Rightarrow (ba)^\omega \leq 0.$$

Proof. We calculate

$$s(ba)^\omega = sba(ba)^\omega \le sbscs(ba)^\omega \le t_1t_2cs(ba)^\omega \le t_1ct_1s(ba)^\omega,$$

whence $s(ba)^\omega \le (t_1ct_2)^\omega$. We calculate further

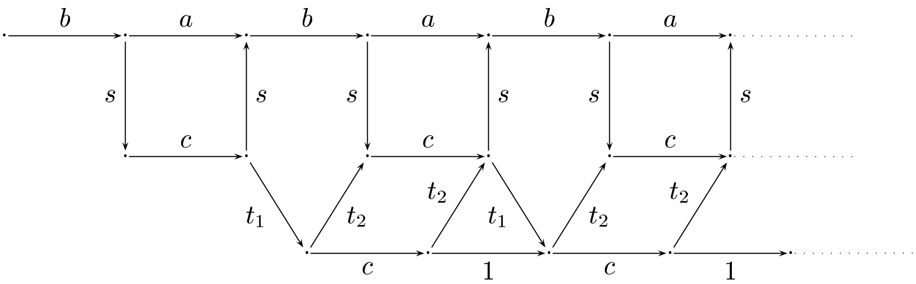
$$ct_2(t_1ct_2)^\omega = ct_2t_1ct_2(t_1ct_2)^\omega \le cct_2(t_1ct_2)^\omega,$$

whence $ct_2(t_1ct_2)^\omega \le c^\omega$. By the assumption $c^\omega \le 0$, therefore

$$(ba)^\omega = ba(ba)^\omega \le bscs(ba)^\omega \le bsc(t_1ct_2)^\omega \le bsc^\omega \le bs0 = 0.$$

$\square$

The proof is visualized by the following diagram. Again the correspondence is immediate.



An extension to further statements in [1] is also straightforward.

Geser has investigated further termination results in presence of commutation properties [10], some of them extending the work of Bachmair and Dershowitz [1] and Bellegarde and Lescanne [2]. The approach with ω-algebra covers some, but not all of these results. See Section 12 for a discussion of its limitations. Geser uses a slight generalization of Bellegarde's and Lescanne's cooperation property with a more commutational flavor. Accordingly, *a b-cooperates* over c, if $c^*a \le b^*a(a+b+c)^*$. As an example, we consider the lemma from p.40 of [10].

Lemma 33. *Let $A \in \mathsf{KA}_\omega$ and let $a, b, c \in A$.*

(i) Let $c^*b^* \le b^*c^*$ and let a b-cooperate over c. Then b^*a type-1 quasi-commutes over $b + c$.

(ii) Under the same conditions as in (i),

$$((b+c)^*a)^\omega \le 0 \Leftrightarrow (b^*a)^\omega \le 0.$$

Proof. (i) We calculate

$$\begin{aligned}(b+c)b^*a &= bb^*a + cb^*a \\ &\leq b^*a + c^*b^*a \\ &\leq b^*a + b^*c^*a \\ &\leq b^*a + b^*b^*a(a+b+c)^* \\ &= b^*a(b^*a+b+c).\end{aligned}$$

(ii) Immediate from (i) and Lemma 24 (ii). □

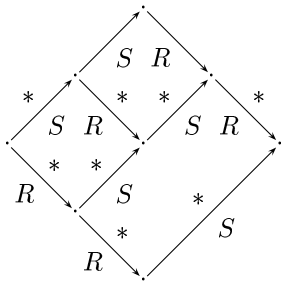
12 Discussion

We have seen that ω-algebra provides rigorous calculational proofs for many interesting statements of abstract rewriting. However, there are limitations. It is not possible to algebraically reconstruct the whole theory of abstract rewriting in ω-algebra. We now characterize the boundary. The iteration algebras we have studied in this text share a particular property. They allow finite or infinite iteration only from the left-hand or right-hand side of a sequence, but not from the interior, that is, progress is made only at the ends of the sequence. At the diagrammatic level this corresponds to arguments where the decomposition of the diagram into use of the induction step and the induction hypothesis takes place at the beginning or end of a diagram, but not at its center. A similar observation holds for coinduction. Formally, Kleene algebra and ω-algebra are not able to deal with simultaneous or nested recursion corresponding to least fixed-points of a mapping like $\lambda x.(axb)$. Note that also a decomposition of such an expression into two nested recursions, one from the left and one from the right, is in general not possible, since the Horn expressions for star induction and omega coinduction cannot be applied with contexts. In other words, ω-algebra captures the regular part of abstract rewriting, but obviously not the context-free one.

A syntactic indication for the limitation therefore are termination assumptions corresponding to both forward and backward traversal of a diagram, which means that relational expressions with and without converse occur in the assumption. The classical example that we cannot treat with ω-algebra is the following variant of Newman's lemma, which has been investigated independently in [14,6] and which is a key theorem in rewriting for pre-congruences.

Lemma 34. *Let R and S be binary relations over some set A. Then $SR \subseteq R^*S^*$ implies $S^*R^* \subseteq R^*S^*$, if $(R \cup S^\circ)$ terminates.*

Here, S° denotes the relational converse of S. The diagrammatic proof is as follows.



The top-most part of the diagram visualizes the induction step. The other parts use the induction hypothesis. The latter diagrams are "smaller", since they can be reached by an R-step or and S°-step from the highest point of the diagram. Reasoning relationally, the induction step is embedded in a context C_1SRC_2 from the left and the right. Such an induction cannot be modeled in ω-algebra. Calculational proofs of (variants of) Newman's lemma have been given in [13,6], however in relation algebras with considerable additional machinery.

Further examples for this limitation can be found, for instance, in [10]. In addition to cooperation, there is a notion of *local cooperation* that uses c instead of c^* in the left-hand side of the defining inequality: *a locally b-cooperates* over c, if $ca \leq b^*a(a+b+c)^*$. Local cooperation implies cooperation in presence of some termination and commutation assumptions (p. 41 of [10]). Again, the induction step in the proof applies at the center of the diagram and the induction hypothesis uses converse. We are not able to give a proof in ω-algebra.

13 Conclusion

This case study shows that ω-algebra, an extension of Kozen's Kleene algebra by infinite iteration, is a useful tool for the termination analysis of programs and rewrite systems. In combination with the results from [16] this supports our general claim that extensions of Kleene algebra provide a reconstruction of and a light-weight formal semantics for a considerable part of diagrammatic reasoning in abstract rewriting. ω-algebra is conceptually simpler than previous approaches; it focuses more on the essence of termination arguments and provides better mechanized proof support and stronger decision procedures. However, our study also shows the limitations. Kleene algebra and ω-algebra can only handle a particular form of "regular" (co)induction that applies at the beginning or end of a rewrite-diagram, but not in the center. This is evident from the particular shape of the Horn identities that occur in their axiomatizations.

We envision the following directions for further work. Kleene algebra with domain [5] provides an alternative modal characterization of termination without an explicit operator for infinite iteration. It seems very interesting to compare both notions of termination, carry over the arguments of this paper to the alternative setting and find out whether similar limitations exist.

We have seen that a proof of Newman's lemma in ω-algebra seems not feasible. Although there is an algebraic proof that uses additional algebraic machinery [6], one might try a leaner calculational proof in Kleene algebra with domain or at least in the μ-calculus, which is more expressive than Kleene algebra with domain but less expressive than relation algebra.

Finally, Kleene algebra is strongly connected with regular languages: The sets of regular languages over some finite alphabet under the regular operations is (up to isomorphism) the free Kleene algebra generated by this alphabet [12]. Therefore, the equational theory of Kleene algebra can be decided by automata. The universal Horn theory of Kleene algebra, however, is undecidable as a consequence of undecidability of the uniform word problem for semigroups [4]. It is very interesting to clarify the relation between ω-algebra, ω-regular languages and automata accepting infinite words. Using tools for automata, large parts of the proofs in this paper could be automated and only some steps involving the application of an induction or coinduction rule might be left for interaction.

Consequently, ω-algebra yields a formal reconstruction of a considerable part of abstract rewriting. It is a simple effective tool for obtaining concise calculational proofs of many interesting statements. The identification of a simple algebra that covers the entire theory, however, remains an interesting open question.

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